Module IV

Fuzzy membership functions, fuzzification, Methods of membership value assignments – intuition – inference – rank ordering, Lambda –cuts for fuzzy sets, Defuzzification methods.

Fuzzy membership functions

Membership function defines the fuzziness in a fuzzy set irrespective of the elements in the set, which arediscrete or continuous. The membership functions are generally represented in graphical form.

There exist certain limitations for the shapes used to represent graphical form of membership function. The rules that describe fuzziness graphically are also fuzzy.

Membership function can be thought of as a technique to solve empirical problems on the basis of experience rather than knowledge. Available histograms and other probability information canalso help in constructingthe membership function.

There are several ways to characterize fuzziness; In a similar way, there are severalways to graphically construct a membership function that describes fuzziness.

Features of the Membership Functions

The membership function defines allthe information contained in a fuzzy set.A fuzzy set A in the universe of discourse *X* can be definedas a set of ordered pairs:

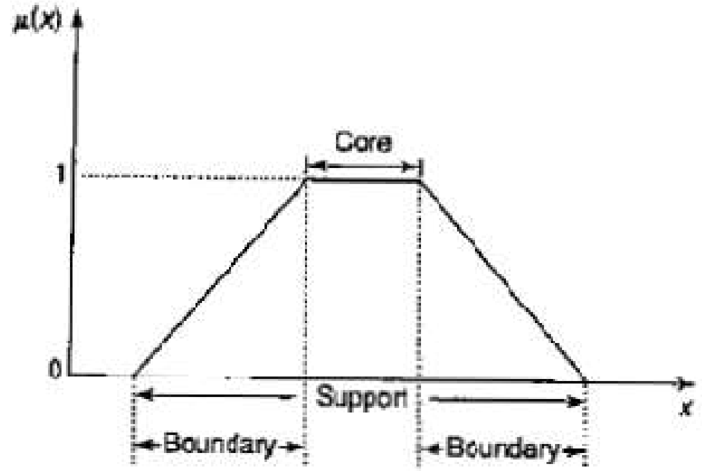
$$\underset{\sim}{A} = \{(x, \mu_{\underset{\sim}{A}}(x)) \mid x \in X\}$$

where $\mu_{\underset{\sim}{A}}(\cdot)$ is called membership function of $\underset{\sim}{A}$.The membership function $\mu_{\underset{\sim}{A}}(\cdot)$ maps *X* to the membershipspace *M,* i.e., $\mu_{\underset{\sim}{A}} : X \rightarrow M$.

The membership value ranges in the interval [0,1], i.e the range of the membership function is a subset of the non-negative real numbers whose supremum is finite .

Basic features of the membership functions

The Figure below shows the basic features of the membership functions. The three main basic features involvedin characterizing membership function are the following.



l. Core: The core of a membership function for some fuzzy set $\underset{\sim}{A}$ is defined as that region of universe is characterized by complete membership in the set $\underset{\sim}{A}$. The core has elements x of the universe such that

$$\mu_{\underset{\sim}{A}}(x) = 1$$

The core of a fuzzy set may be an empty set.

2. **Support:**The support of a membership function for a fuzzy set $\underset{\sim}{A}$is defined as that region of universe thatis characterized by a nonzero membership in the set $\underset{\sim}{A}$. The support comprises elements x from the universesuch that

$$\mu_{\underset{\sim}{A}}(x) > 0$$

A fuzzy set whose support is a single element in Xwith$\mu_{\underset{\sim}{A}}(x) = 1$is referred to as a fuzzy singleton.

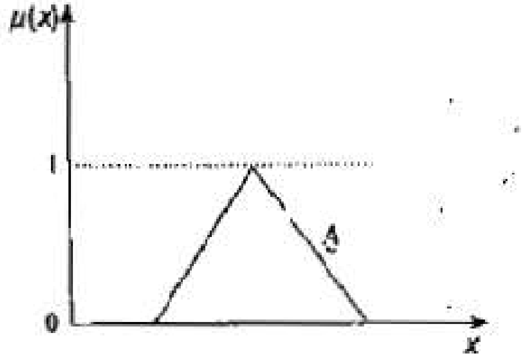
3.Boundary: The support of a membership function for a fuzzy set $\underset{\sim}{A}$ is defined as that region of universecontaining elements that have a non zero but not complete membership. The boundary comprises thoseelements of x of the universe such that

$$0 < \mu_{\underset{\sim}{A}}(x) < 1$$

Types of fuzzy set

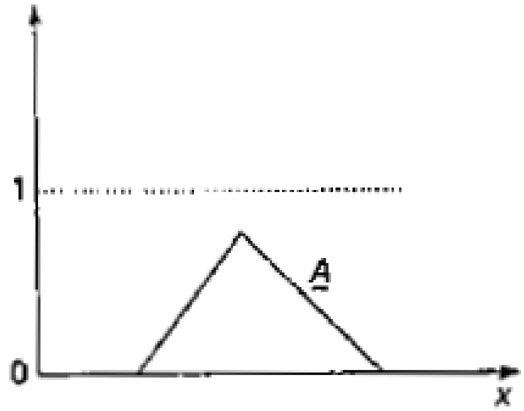
1) **Normal Fuzzy set**

A fuzzy set whose membership function has at least one elementx in the universe whose membership value is unity is called**normal fuzzy set**. The element for which the membership is equal to 1 is called prototypicalelement.



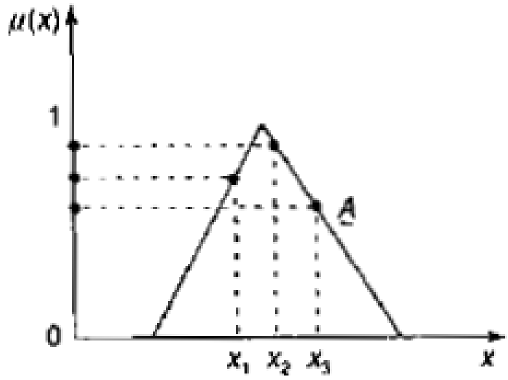
2) **Subnormal Fuzzy set**

A fuzzy set where in no membershipfunction has its value equal to 1 is called **subnormal fuzzy set**.



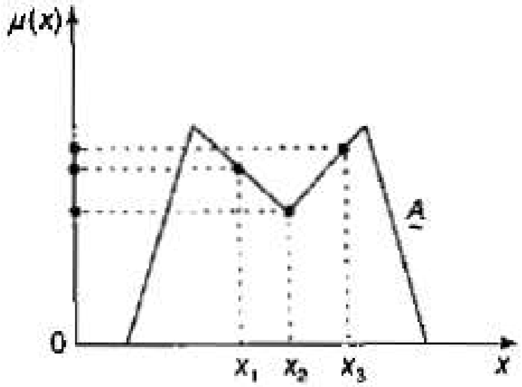
3) Convex Fuzzy set

A *convex fuzzy set* has a membership function whose membership values are strictly monotonically increasingor strictly monotonically decreasing or strictly monotonically increasing than strictly monotonically decreasingwithincreasing values for elements in the universe.



4) Nonconvex Fuzzy set

Afuzzy set possessing characteristics opposite to that of convex fuzzy set is called **nonconvexfuzzyset**, i.e., the membership values of the membership functionare not strictly monotonically increasing or decreasing or strictly monotonically increasing than decreasing.



Fuzzification

Fuzzification is the process of transforming a crisp set to a fuzzy set or a fuzzy set to a fuzzier set, i.e.,crisp quantities are converted to fuzzy quantities. This operation translates accurate crisp input values into linguistic variables.

A Fuzzy set $\underset{\sim}{A} = \{\mu_i / x_i | x_i \in X\}$, a common fuzzification algorithm is performed by keeping μ_i constant and x_i being transformed to a fuzzy set $Q(x_i)$ depicting the expression about x_i. The fuzzy set $Q(x_i)$ is referred to as the kernel of fuzzification. The fuzzified set A can be expressed as

$$\underset{\sim}{A} = \mu_1 Q(x_1) + \mu_2 Q(x_2) + \cdots + \mu_n Q(x_n)$$

where the symbol ~ means fuzzified.

This process of fuzzification is called support fuzzification (s-fuzzification). There is another method of fuzzification called *grade fuzzification* (g-fuzzification) where X_i is kept constant and μ_i is expressed as a fuzzy set. Thus, using these methods, fuzzification is carried out.

Methods of Membership Value Assignments

The method of assigning membership values are as follows:

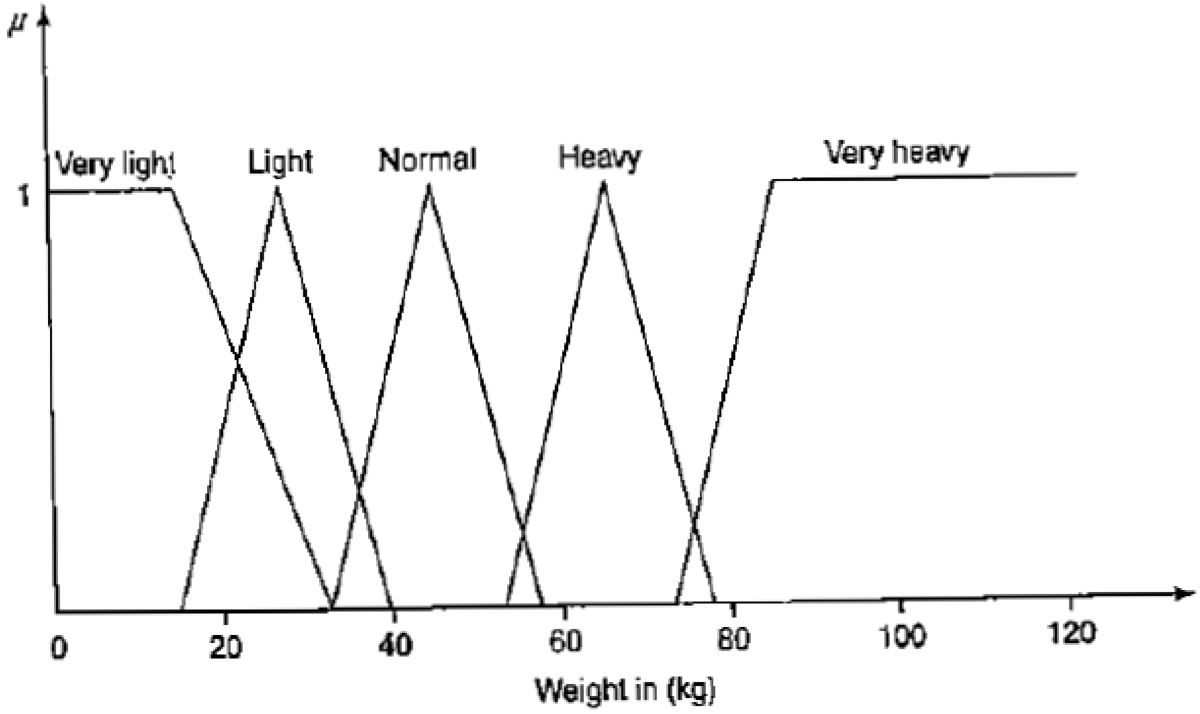
1. Intuition
2. Inference
3. Rank Ordering
4. Angular Fuzzy Sets
5. Neural Networks
6. Genetic Algorithm
7. Inductive Reasoning.

1.Intuition

Intuition method is based upon the common intelligence ofhuman. It is the capacity of the human to developmembership functions on the basis of their own intelligence and understanding capacity. There should be an in-depth knowledge of the application to whichmembership value assignment has to be made.

The Figure below shows various shapes of weights of people measured in kilogram in the universe. Each curve is a membershipfunction corresponding to various fuzzy (linguistic) variables; such as very light, light, normal, heavy andvery heavy.

The curves are based on context functions and the human developing them. For example, if theweights are referred to range of thin persons we get one set of curves, and if they are referred to range ofnormal weighing persons we get another set and so on. The main characteristics of these curves for their usagein fuzzy operationsare based on their overlapping capacity.



2)Inference

The inference method *uses* knowledge to perform deductive reasoning. Deduction achieves conclusion bymeans of forward inference. There are various methods for performing deductive reasoning.

Here the knowledgeof geometrical shapes and geometry is used for defining membership values. The membership functionsmay be defined by various shapes: triangular, trapezoidal, bell-shaped, Gaussian and so on. The inferencemethod here is discussed via triangular shape.

Consider a triangle, where *X, Y* and *Z* are the angles such that $X \geq Y \geq Z \geq 0$ and let *u* be the universe of triangles, i.e.,

$$U = \{(X, Y, Z) | X > Y \geq Z \geq 0; X + Y + Z = 180\}$$

There are various types of triangles available. Here a few are considered to explain inference methodology:

$\underset{\sim}{I}$ = isosceles triangle (approximate)
$\underset{\sim}{E}$ = equilateral triangle (approximate)
$\underset{\sim}{R}$ = right-angle triangle (approximate)
$\underset{\sim}{IR}$ = isosceles and right-angle triangle (approximate)
$\underset{\sim}{T}$ = other triangles

The membership values of approximate isosceles triangle is obtained using the following definition, where: $X \geq Y \geq Z \geq 0$ and $X + Y + Z = 180°$:

$$\mu_{\underset{\sim}{I}}(X, Y, Z) = 1 - \frac{1}{60°} \min(X - Y, Y - Z)$$

If X=Y or Y=Z and if X=120º,Y=60º and Z=0º , we get

$$\begin{aligned} \mu_{\underset{\sim}{I}}(X, Y, Z) &= 1 - \frac{1}{60°} \min(120° - 60°, 60° - 0°) \\ &= 1 - \frac{1}{60°} \min(60°, 60°) \\ &= 1 - \frac{1}{60°} \times 60° \\ &= 1 - 1 = 0 \end{aligned}$$

The membership value of approximate right-angle triangle is given by·

$$\mu_{\underset{\sim}{R}}(X, Y, Z) = 1 - \frac{1}{90°} |X - 90°|$$

If X= 90°, the membership value of a right-angle triangle is1, andif X= 180º,the membership value $\mu_{\underset{\sim}{R}}$ becomes 0:

$$X = 90^\circ \Rightarrow \mu_{\underset{\sim}{R}} = 1$$
$$X = 180^\circ \Rightarrow \mu_{\underset{\sim}{R}} = 0$$

The membership value of approximate isosceles right angle triangle is obtained by raking the logical intersection of the approximate isosceles and approximate right-angle triangle membership function.i.e.,

$$\underset{\sim}{IR} = \underset{\sim}{I} \cap \underset{\sim}{R}$$

and it is given by,

$$\begin{aligned}\mu_{\underset{\sim}{IR}}(X, Y, Z) &= \min[\mu_{\underset{\sim}{I}}(X, Y, Z), \mu_{\underset{\sim}{R}}(X, Y, Z)] \\ &= 1 - \max\left[\frac{1}{60^\circ}\min(X - Y, Y - Z), \frac{1}{90^\circ}|X - 90^\circ|\right]\end{aligned}$$

The membership function for a fuzzy equilateral triangle is given by

$$\mu_{\underset{\sim}{E}}(X, Y, Z) = 1 - \frac{1}{180^\circ}|X - Z|$$

The membership function of other triangles, denoted by, $\underset{\sim}{T}$ is the complement of the. logical union of $\underset{\sim}{I}$ $\underset{\sim}{R}$ and $\underset{\sim}{E}$ i.e.,

$$\underset{\sim}{T} = \overline{\underset{\sim}{I} \cup \underset{\sim}{R} \cup \underset{\sim}{E}}$$

By using De Morgan's law, we get

$$\underset{\sim}{T} = \bar{\underset{\sim}{I}} \cap \bar{\underset{\sim}{R}} \cap \bar{\underset{\sim}{E}}$$

The membership value can be obtained using the equation

$$\mu_T(X,Y,Z) = \min\{1-\mu_I(X,Y,Z), 1-\mu_E(X,Y,Z), 1-\mu_R(X,Y,Z)\}$$
$$= \frac{1}{180^\circ}\min\{3(X-Y), 3(Y-Z), 2|X-90^\circ|, X-Z\}$$

Rank Ordering

The formation of government is based on the polling concept; to identify a best student, ranking may beperformed; to buy a car, one can ask for several opinions and so on.

All the above mentioned activities are carriedout on the basis of the preferences made by an individual, a committee, a poll and other opinion methods.

This methodology can be adapted to assign membership values to a fuzzy variable. Pairwise comparisons enable us to determine preferences and this results in determining the order of the membership.

Lambda –cuts for fuzzy sets (Alpha-Cuts)

Consider a fuzzy set $\underset{\sim}{A}$. The set $A_\lambda (0 < \lambda < 1)$,called the lambda (λ)- cut or (alpha cut [α]-cut)set, is a crisp set of the fuzzy set and is defined as follows:

$$A_\lambda = \{x \mid \mu_{\underset{\sim}{A}}(x) \geq \lambda\}; \quad \lambda \in [0,1]$$

The setA$_\lambda$.is called a weak lambda-cut set if it consists of all the elements of a fuzzy set whose membershipfunctions have values greater than or equal to a specified value. On the other hand, the setA$_\lambda$, is called a Strong lambda-cut set if it consists ofall the elements of a fuzzy set whose membership functions have values strictlygreater than specified value. A strong λ-cut set is given by

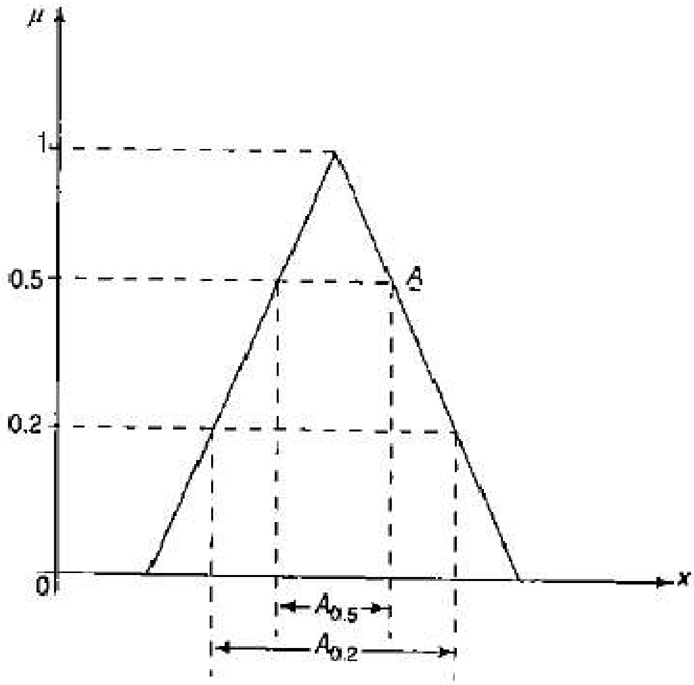
$$A_\lambda = \{x \mid \mu_{\underset{\sim}{A}}(x) > \lambda\}; \quad \lambda \in [0,1]$$

The properties of λ-cut sets are as follows

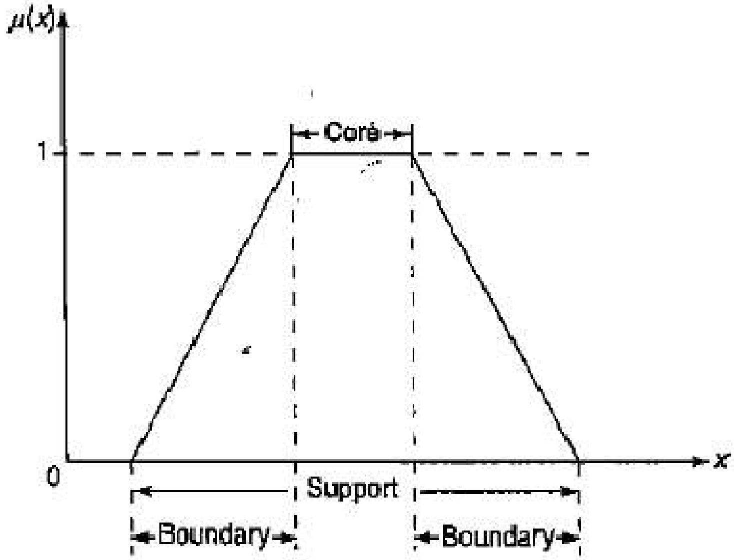
1. $(\underset{\sim}{A} \cup \underset{\sim}{B})_\lambda = A_\lambda \cup B_\lambda$
2. $(\underset{\sim}{A} \cap \underset{\sim}{B})_\lambda = A_\lambda \cap B_\lambda$

3. $(\overline{\underset{\sim}{A}})_\lambda \neq (\overline{A}_\lambda)$ except when $\lambda = 0.5$
4. For any $\lambda \leq \beta$, where $0 \leq \beta \leq 1$, it is true that $A_\beta \subseteq A_\lambda$, where $A_0 = X$.

The fourth property is essentially used in graphics. The figure below shows a continuousvalued fuzzy with two λ-cut values.



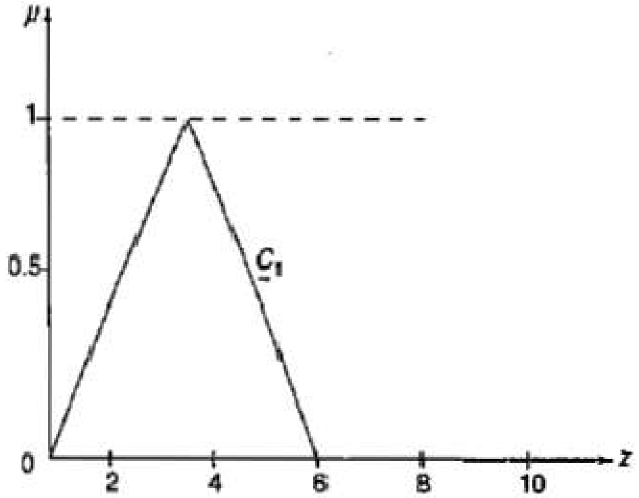
The Figure belowshowsthe features of the membership functions.



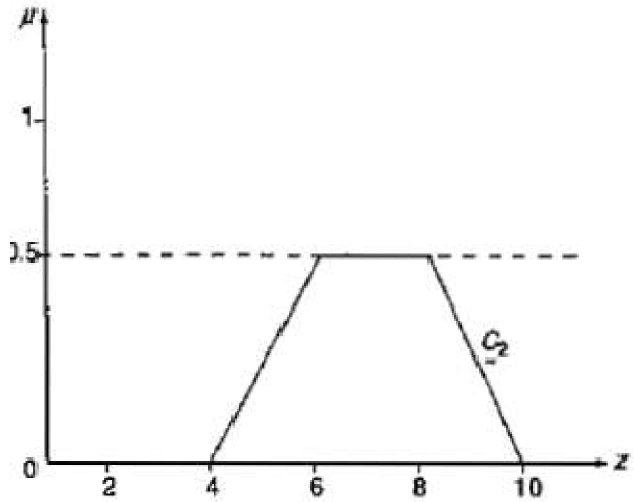
Defuzzification Methods

Defuzzification isthe process of conversion of a fuzzy quantity into a precise quantity. The output of a fuzzyprocess may be union of two or more fuzzy membership functions defined on the universe of discourse of theoutputvariable.

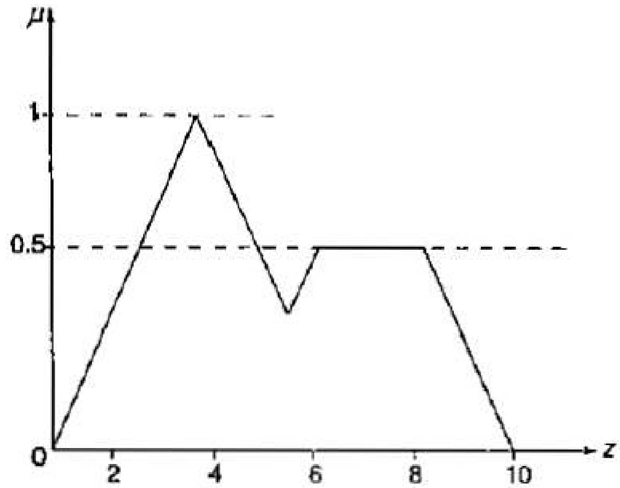
Consider a fuzzy output comprising two parts: the first part, $\underset{\sim}{C}_1$, a triangular membership shape as shown in following Figure:



The second part, C_2, a trapezoidal shape as shown in Figure below:



The union of these two membership functions, *i.e.,* $C = C_1 \cup C_2$ involves the max operatorwhich is going to be the outer envelope of the two shapes shown in Figures above. The final shape of C is shown in figure below:



A fuzzy output process may involve many output parts, and the membership function representingeachpart of the output can have any shape. The membership function of the fuzzy output need not always be normal. In general,

$$\underset{\sim}{C}_n = \bigcup_{i=1}^{n} \underset{\sim}{C}_i = \underset{\sim}{C}$$

Defuzzification methods include the following:

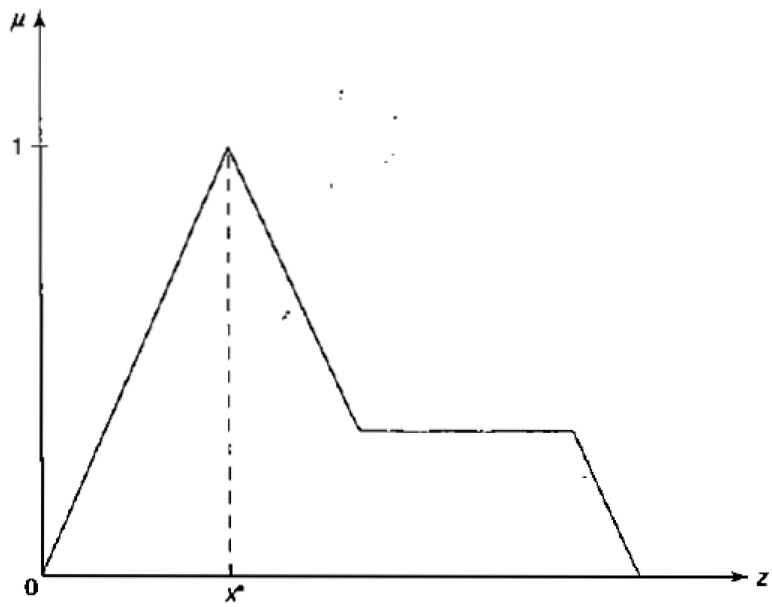
1. Max-membership principle.
2. Centroid method
3. Weighted average method.
4. Mean-max membership.
5. Center of sums.
6. Center oflargest area.
7. First of maxima, last of maxima.

1) Max-Membership Principle

This method is also known as height method and is limited to peak output functions. This method is givenby the algebraic expression

$$\mu_{\underset{\sim}{C}}(x^*) \geq \mu_{\underset{\sim}{C}}(x) \text{ for all } x \in X$$

The method is illustrated in Figure below:

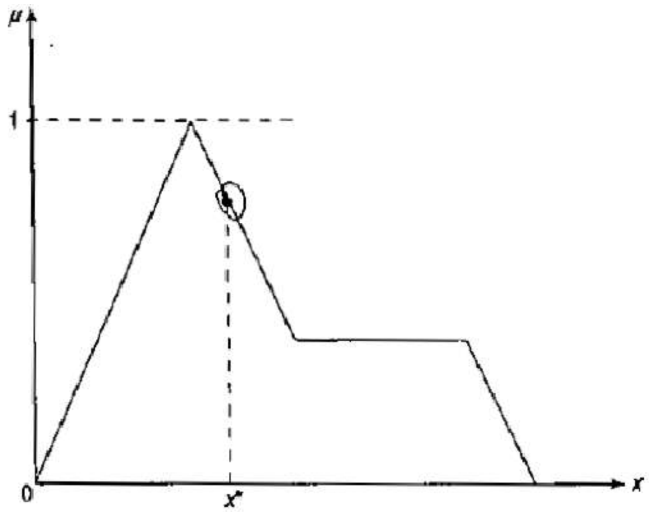


2) Centroid Method

This method is also known as center of mass, center of area or center of gravity method.It is the mostcommonly used defuzzification method. The defuzzified output $x*$ is defined as

$$x^* = \frac{\int \mu_C(x) \cdot x dx}{\int \mu_C(x) dx}$$

where the symbol $\int$ denotes an algebraic integration. This method is illustrated inFigure below:



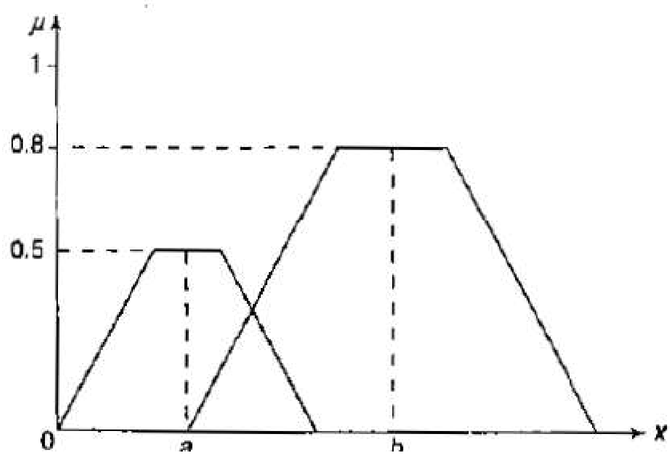
3) Weighted Average Method

This method is valid for symmetrical output membershipfunction only. Each membership function is weighted by its maximum membership value. The output in this cas is given by,

$$x^* = \frac{\sum \mu_C(\bar{x}_i) \cdot \bar{x}_i}{\sum \mu_C(\bar{x}_i)}$$

whereΣdenotes algebraic sum and $\bar{x}_i$is the maximum of the ithmembership function. The methodis illustrated in Figure below, where two fuzzy sets are considered. From the Figure, we notice that thedefuzzified output is given by

$$x^* = \frac{0.5a + 0.8b}{0.5 + 0.8}$$

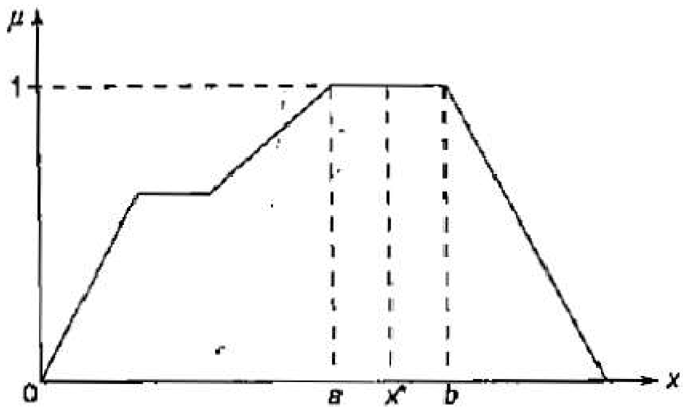


4) Mean-Max Membership

This method is also known as the middle of the maxima. This is closely related to max-membershipmethod, except that the locations of the maximum membership can be nonunique. The output here isgiven by

$$x^* = \frac{\sum_{i=1}^{n} \bar{x}_i}{n}$$

This is illustrated in Figure below:



From Figure above, we notice that the defuzzified output is given by,

$$x^* = \frac{a+b}{2}$$

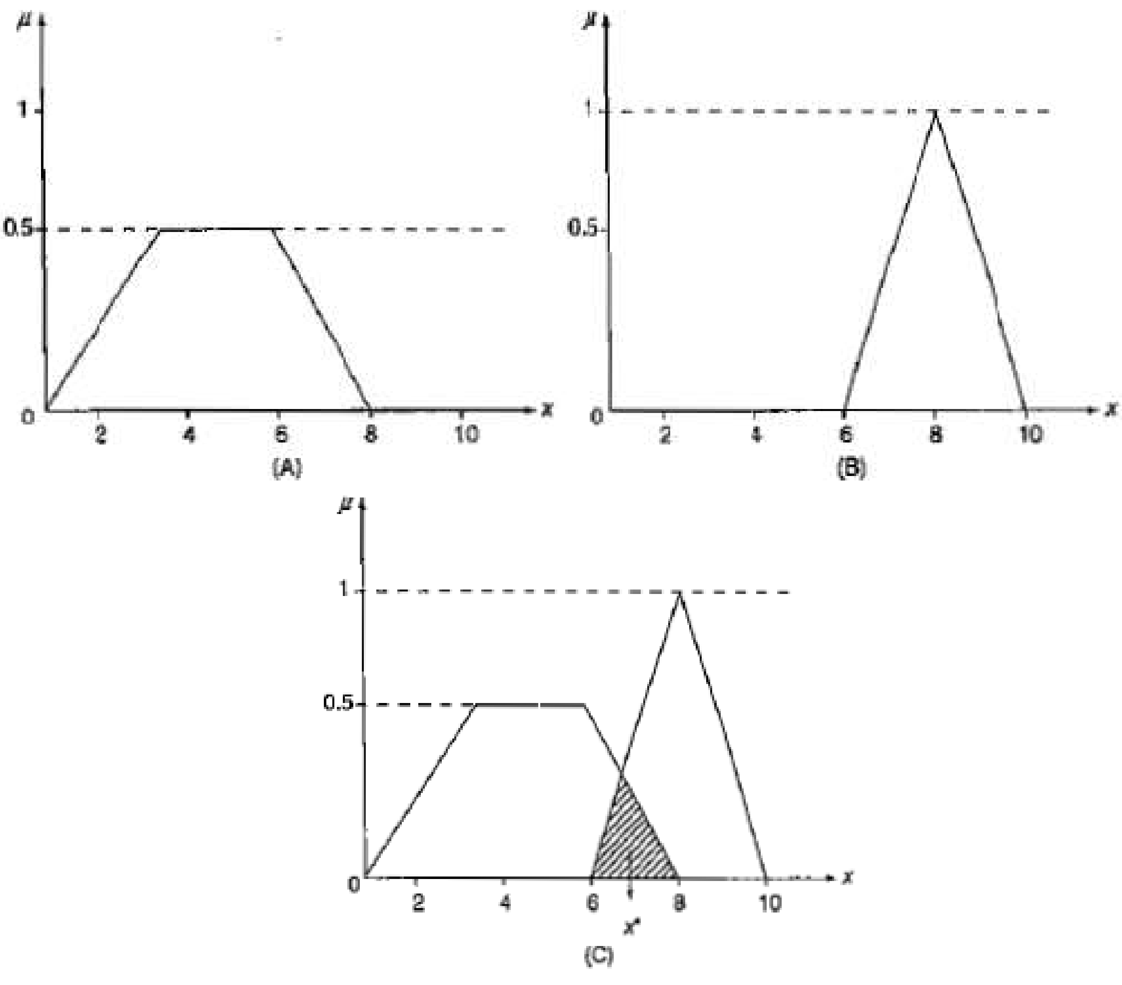
where*a* and *b* are as shown in the figure.

5) Center of Sums

This method employs the algebraic sum of the individual fuzzy subsets instead oftheir union. The calculations here are very fast, but the main drawback is that intersectingareas are added twice. The defuzzified value $x*$ is given by

$$x^* = \frac{\int_x x \sum_{i=1}^{n} \mu_{C_i}(x)dx}{\int_x \sum_{i=1}^{n} \mu_{C_i}(x)dx}$$

The Figure below illustrates the center of sums method. In center of sums method, the weights are the areas ofthe respective membership functions, whereas in the weighted average method the weights are individual membership values.



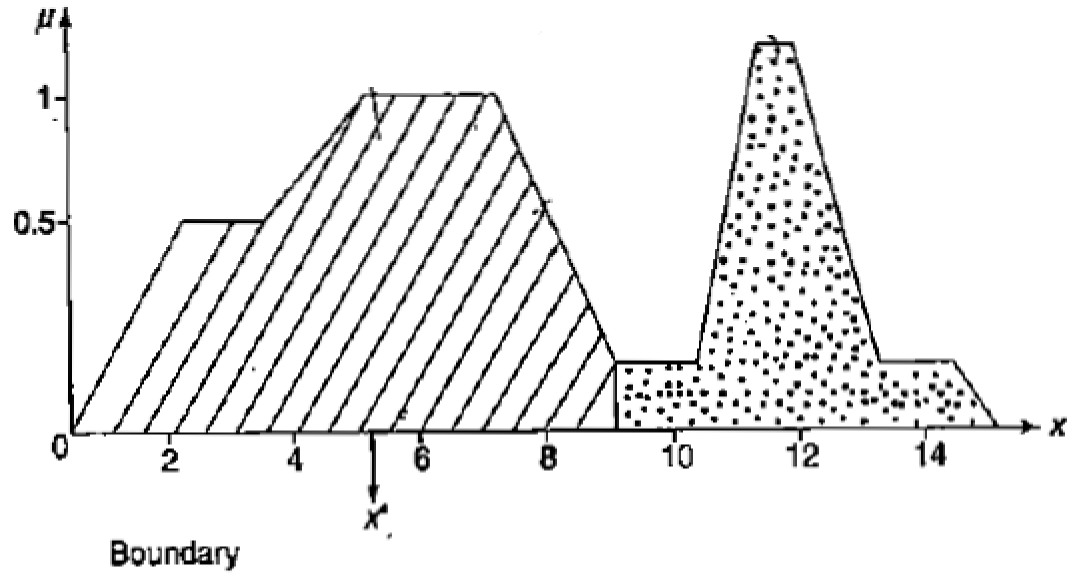
[A) First and (B) second membership functions, (C) defuzzification.

6) **Center of Largest Area**

This method can be adopted when the output consists of at least two convex fuzzy subsets which are not overlapping. The output in this case is biased towards side of one membership function. When output fuzzyset has at least two convex regions, then the center of gravity ofthe convex fuzzysubregionhaving the largest area is used to obtain the defuzzified value x*.This value is given by

$$x^* = \frac{\int \mu_{\underline{C}_j}(x) \cdot x dx}{\int \mu_{\underline{C}_j}(x) dx}$$

Where $\underline{C}_j$ is the convex subregion that has the largest area making up $\underline{C}_j$ .Figure below illustrates the center oflargest area.



7) **First of Maxima (Last of Maxima)**

This method uses the overall output or union of all individual output fuzzysets $\underline{C}_j$ for determining the smallest value of the domain with maximized membershipin $\underline{C}_j$ .The steps used for obtaining x*areas follows:

1. Initially, the maximum height in the union is found:

$$\text{hgt}(\underline{C}_j) = \sup_{x \in X} \mu_{\underline{C}_j}(x)$$

where sup is supremum, i.e., the least upperbound:

$$x^* = \inf_{x \in X} \left\{ x \in X \middle| \mu_{\underline{C}_j}(x) = \text{hgt}(\underline{C}_j) \right\}$$

2. Then the first of maxima is found:

$$x^* = \inf_{x \in X} \{ x \in X \mid \mu_{C_i}(x) = \mathrm{hgt}(C_i) \}$$

whereinf is the minimum, i.e., the greatest lower bound.

3. After this the last maxima is found:

$$x^* = \sup_{x \in X} \{ x \in X \mid \mu_{C_i}(x) = \mathrm{hgt}(C_i) \}$$

where sup= supremum, i.e., the least upperbound; inf =infimum, i.e., the greatest lower bound. This is illustrated in Figure below. From Figure below, the first maxima is also the last maxima, and since it isa distinct max, it is also the mean-max.